FRACTALS-8

**Multifractal time series**

Many records do not exhibit a simple monofractal scaling behavior, which can be accounted for by a single scaling exponent. As discussed in the previous sections, there might exist crossover (time-) scales  separating regimes with different scaling exponents. In other cases, the scaling behavior is more complicated, and different scaling exponents are required for different parts of the series. In even more complicated cases, such different scaling behavior can be observed for many interwoven fractal subsets of the time series. In this case a multitude of scaling exponents is required for a full description of the scaling behavior in the same range of time scales, and a multifractal analysis must be applied. Multifractal scaling is observed if the scaling behaviour of small and large fluctuations is different. For example, extreme events might be more or less correlated than typical events.

*Multifractal Detrended Fluctuation Analysis*

While fractal processes are characterized by a long-term correlations and single scaling exponent, in multifractal time series, subsets with small and large fluctuations scale differently, and their description requires a hierarchy of scaling exponents [1].

Several methods have been developed for multifractal analysis of non stationary signals:

* Wavelet transform modulus maxima (WTMM) method [2]
* Multifractal detrended fluctuation analysis (MF-DFA) method [3]
* Multifractal detrending moving average method (MF-DMA) [4]

It was shown that MF-DFA method produces more reliable results than WTMM method [5], and has been used to analyze physiological signals [6,7], geophysical data [8,9,10], weather data [11,12], hydrological records [13,14] and financial time series [15,16,17].

The implementation of MF-DFA algorithm is described as follows [3].

1. The first step is integration of original series  to produce



where is the average.

1. Next, the integrated series  is divided into  non-overlapping segments of length  and in each segment the local trend (linear or higher order polynomial least square fit) is estimated and subtracted from.
2. The detrended variance



is calculated for each segment and then averaged over all segments to obtain th order fluctuation function



where, in general,  can take any real value except zero.

1. Repeating this calculation for all box sizes provides the relationship between fluctuation function  and box size. If long-term correlations are present,  increases with  according to a power law

.

The scaling exponent  is obtained as the slope of the linear regression of  versus.

For  

For stationary time series,  is identical to the well-known Hurst exponent , and therefore  is called the generalized Hurst exponent. For positive values of  the generalized Hurst exponent  describes the scaling behavior of large fluctuations, while for negative values of ,  describes the scaling behavior of small fluctuations. For monofractal time series  is independent of , while for multifractal time series small and large fluctuations scale differently and  is a decreasing function of .

Generalized Hurst exponents  are related to the Renyi exponents  defined by the standard partition function-based multifractal formalism

.

For monofractal signals  is linear function of  (as), and for multifractal signals  is nonlinear function of .

Another way to characterize multifractal process is the singularity spectrum  which is related to  through the Legendre transform

 

where  is the fractal dimension of the support of singularities in the measure with Lipschitz-Holder exponent . The singularity spectrum of monofractal signal is represented by a single point in the  plane, whereas multifractal process yields a single humped function.

Two different types of multifractality in time series can be distinguished, both requiring a hierarchy of scaling exponents: a) multifractality can be due to a broad probability density function for the values of the time series; and b) multifractality can also be due to different long-term correlations for small and large fluctuations.

The type of multifractality can be determined by analyzing the corresponding randomly shuffled series [3].

In the case of multifractals of type b) the shuffling procedure destroys correlations and shuffled series exhibit simple random behavior with . For multifractals of type a) the original dependence is not changed , since the multifractality is due to the probability density, which is not affected by the shuffling procedure. If both kinds of multifractality are present in a given series, shuffled series exhibit weaker multifractality than the original one.

Multifractal spectrum reflects the level of complexity of the underlying stochastic process, that can be characterized by a set of three parameters, as proposed by Shimizu et al. [18]. First, regression of  is performed to the quadratic function



The first parameter  is the position of the maximum of the curve. The second parameter  is an asymmetry parameter, which is zero for symmetric shapes, and positive (negative) for left- (right-) skewed (centered) shapes. The third parameter is the width of the spectrum  that is obtained by extrapolating the fitted curve to zero, where  and  are the Holder exponents of the strongest and weakest singularity respectively.

These three parameters can be used as a measure of complexity of the process. A small value of means that the underlying process is correlated and more regular in appearance. The width  of the spectrum measures the degree of multifractality of the process (the wider the range of the fractal exponents, the “richer” the structure of the process). The asymmetry parameter indicates which fractal exponents are dominant. If high fractal exponents are dominant,  spectrum is right-skewed  and the process is characterized by “fine structure”(small fluctuations). If low fractal exponents are dominant, the process is more regular or smooth, and  spectrum is left-skewed (large fluctuations).

In summary, a signal with a high value of , a wide range  of fractal exponents (higher degree of multifractality), and a right-skewed shape  may be considered more complex than those with opposite characteristics [18].

In some cases the quadratic fit does not explain well the observed  spectrum, and a fourth-degree polynomial



can be used. The asymmetry which depends on the first order coefficient  and the third order coefficient  , can be quantified by the ratio  , where  for symmetric shapes,  for right skewed shapes, and  for left skewed shapes [19].

**Example** (Adapted from Ref. [20])

We apply MFDFA on daily hot-pixels data detected in Brazil by the satellite NOAA-12 during the period 1998-2006.

 

Figure 1. Left: Number of hot pixels per day detected by NOAA-12 between 1998 and 2006. Right: Yearly histogram of hot-pixels detected by NOAA-12 between 1998 and 2006.



Figure 2. Fluctuation function  versus box size  on double logarithmic scale, for different values of from -10 to 10 with a step of 1 (from bottom, to top).



Figure 3. Left: Generalized Hurst exponent  before and after shuffling of the data. Right: The spectrum for the two periods, before and after July 2002. The full lines represent regression curves to the fourth order polynomial form .

Table 1: Complexity parameters , and for the data detected by NOAA-12 between 1998 and 2006, and the two time periods before and after July 2002

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
| All data | 0.8638 | 0.6136 | 1.247 |
| 1998-2002 | 0.8383 | 0.7176 | 1.138 |
| 2002-2006 | 0.8505 | 0.4541 | 1.270 |

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